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State-Dependent Riccati Equation for Control of Aeroelastic Flutter

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Feedback control design for suppression of flutter in an aeroelastic system is studied. Feedback control laws based on the state-dependent Riccati equation are derived. Two control laws are studied: one formulation uses an observer to estimate the states after which it uses the estimated states to generate feedback law; the other formulation uses partial observation to directly generate the control law.

I. Introduction

WE study the effectiveness of a feedback control law in controlling the instability of an airfoil in flutter conditions. Aeroelastic instability can arise when an elastic structure is moving at a high speed in a surrounding fluid. The coupling between the aerodynamic forces and elastic forces can lead to instabilities, which can cause structural damage (Ref. 1, p. 51). One class of problems deals with the characterization and identification of the flutter speed above which flutter instabilities set in.[2] Another class of problems deals with the design of active control strategies to damp out the vibrations due to flutter conditions. A number of investigators have considered control problems for such systems.[3,4] Recent results include the application of neural networks,[5] control of a nonlinear aeroelastic system based on Euler–Lagrange theory,[6] and output feedback linearization.[7,8]

The purpose of this work is to consider the effectiveness of a nonlinear controller based on the state-dependent Riccati equation (SDRE). The method of SDRE has recently been used for control problems involving nonlinear systems.[9,10] This theory is motivated by the fact that, for a linear system, the linear-quadratic-regulator (LQR) approach can be effectively used to obtain control laws. For a nonlinear system a form of linearization is required that is not an approximation but simply a rewriting of the mathematical model in a different form. This form is often called apparent linearization.[11] It is then possible to obtain feedback control laws that are similar to LQR theory in computation. Section II presents the problem in detail and casts the working equations in the apparent linearization form. Section III formulates the feedback control law based on SDRE. Section IV considers an estimator for the current system based on SDRE, and Sec. V investigates the performance of the control design based on SDRE.

II. Aeroelastic System and the Open-Loop Problem

Consider the cross section of a wing shown in Fig. 1. The governing equations of motion are given by[8,12]

$$\begin{bmatrix} m & m\chi_\alpha b \\ m\chi_\alpha b & I_\alpha \end{bmatrix}\begin{bmatrix} \ddot h \\ \ddot\alpha \end{bmatrix} + \begin{bmatrix} c_h & 0 \\ 0 & c_\alpha \end{bmatrix}\begin{bmatrix} \dot h \\ \dot\alpha \end{bmatrix} + \begin{bmatrix} k_h & 0 \\ 0 & k_\alpha(\alpha) \end{bmatrix}\begin{bmatrix} h \\ \alpha \end{bmatrix} = \begin{bmatrix} -L \\ M \end{bmatrix} \tag{1}$$

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where h is the plunge motion and α is the pitch angle. The parameter m is the mass of the wing, and c_α and c_h are pitch and plunge structural damping coefficients, respectively. The parameters L and M are the aerodynamic lift and moment. Assuming a quasi-steady aerodynamic model, the aerodynamic lift and moment are given by

$$L = \rho U^2 b c_{l_\alpha}\left[\alpha + \dot h/U + \left(\tfrac{1}{2} - a\right) b(\dot\alpha/U)\right] + \rho U^2 b c_{l_\beta}\beta \tag{2}$$

$$M = \rho U^2 b c_{m_\alpha}\left[\alpha + \dot h/U + \left(\tfrac{1}{2} - a\right) b(\dot\alpha/U)\right] + \rho U^2 b c_{m_\beta}\beta \tag{3}$$

where c_{l_α} and c_{m_α} are lift and moment coefficient per angle of attack, respectively, and c_{l_β} and c_{m_β} are lift and moment coefficient per unit control surface deflaction, respectively. In this note the stiffness parameter $k_\alpha(\alpha)$ is modeled by a nonlinear polynomial fit given by[12]

$$k_\alpha(\alpha) = 2.82(1 - 22.1\alpha + 1315.5\alpha^2 - 8580\alpha^3 + 17289.7\alpha^4) \tag{4}$$

Let the vector $\boldsymbol{x} \in R^4$ be given by $\boldsymbol{x} = [h, \alpha, \dot h, \dot\alpha]^\top$; then the preceding equations can be written in a state-space form given by

$$\dot{\boldsymbol{x}} = \boldsymbol{A}(\boldsymbol{x})\boldsymbol{x} + \boldsymbol{b}\beta(t) \tag{5}$$

where

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k_1 & -[k_2U^2 + p(x_2)] & -c_1 & -c_2 \\ -k_3 & -[k_4U^2 + q(x_2)] & -c_3 & -c_4 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} 0 \\ 0 \\ g_3U^2 \\ g_4U^2 \end{bmatrix} \tag{6}$$

and where the parameters are given as follows:

$$d = m\left(I_\alpha - m\chi_\alpha^2 b^2\right)$$

$$k_1 = I_\alpha k_h/d, \qquad k_2 = \left(I_\alpha \rho b c_{l_\alpha} + m\chi b^3 \rho c_{m_\alpha}\right)/d$$

$$k_3 = -m\chi_\alpha b k_h/d, \qquad k_4 = -\left(m\chi b^2 \rho c_{l_\alpha} + m\rho b^2 c_{m_\alpha}\right)/d$$

$$p(x_2) = -m\chi_\alpha b k_\alpha(x_2)/d, \qquad q(x_2) = m k_\alpha(x_2)/d,$$

$$c_1 = \left[I_\alpha\left(c_h + \rho U b c_{l_\alpha}\right) + m\chi_\alpha \rho U b^3 c_{m_\alpha}\right]/d$$

$$c_2 = \left[I_\alpha \rho U b^2 c_{l_\alpha}\left(\tfrac{1}{2} - a\right) - m\chi_\alpha b c_\alpha + m\chi_\alpha \rho U b^4 c_{m_\alpha}\left(\tfrac{1}{2} - a\right)\right]/d$$

$$c_3 = -m\left(\chi_\alpha b c_h + \chi_\alpha \rho U b^2 c_{l_\alpha} + \rho U b^2 c_{l_\alpha}\right)/d$$

$$c_4 = m\left[c_\alpha - \chi_\alpha \rho U b^3 c_{l_\alpha}\left(\tfrac{1}{2} - a\right) - \rho U b^3 c_{m_\alpha}\left(\tfrac{1}{2} - a\right)\right]/d$$

$$g_3 = -\left(I_\alpha \rho b c_{l_\beta} + m\chi_\alpha b^3 \rho c_{m_\beta}\right)/d$$

$$g_4 = \left(m\chi_\alpha b^2 \rho c_{l_\beta} + m\rho b^2 c_{m_\beta}\right)/d$$

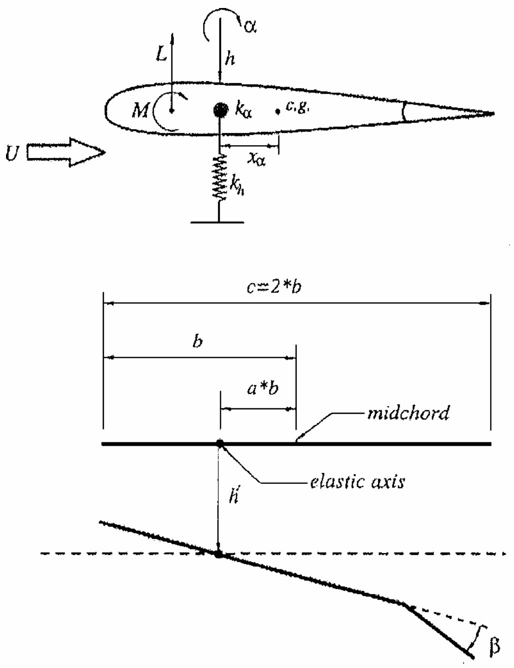

Fig. 1 Wing cross section.

It is well known that the preceding open-loop system can exhibit unstable behavior for various flow conditions U and a including limit cycles.[12,13]

III. Nonlinear Feedback Control Law

Assume that the full state is available for feedback and consider the optimal infinite-horizon regulator problem of the following form. Minimize

$$J = \frac{1}{2}\int_0^\infty [\boldsymbol{x}^\top \boldsymbol{x} + r u^2]\,\mathrm{d}t \tag{7}$$

subject to the system constraint

$$\dot{\boldsymbol{x}} = \boldsymbol{A}(\boldsymbol{x})\boldsymbol{x} + \boldsymbol{b}u(t), \qquad u(t) = \beta(t) \tag{8}$$

It follows that $\boldsymbol{A}(\boldsymbol{x})\boldsymbol{x}|_{x=0} = 0$ and $\boldsymbol{b}|_{x=0} \neq 0$. The SDRE approach for obtaining a suboptimal solution of the preceding problem is to solve the alebraic Riccati equation given by

$$\boldsymbol{A}^\top(\boldsymbol{x})\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}(\boldsymbol{x}) - \boldsymbol{P}\boldsymbol{b}r^{-1}\boldsymbol{b}^\top\boldsymbol{P} + \boldsymbol{I} = 0 \tag{9}$$

for the matrix $\boldsymbol{P} \geq 0$. The feedback control law is then given by

$$u(t) = -r^{-1}\boldsymbol{b}^\top \boldsymbol{P}(\boldsymbol{x})\boldsymbol{x} \tag{10}$$

For a linear time-invariant system, that is, $\boldsymbol{A} = \text{const}$, the preceding state feedback law is guaranteed to have a solution if the pair $(\boldsymbol{A}, \boldsymbol{b})$ is stabilizable. For the nonlinear case we have Theorem 1.[10]

Theorem 1: Assume that columns of $\boldsymbol{A}(\boldsymbol{x}) \in C^1$ in the neighborhood Ω about the origin and that the pair $(\boldsymbol{A}(\boldsymbol{x}), \boldsymbol{b})$ is pointwise stabilizable in the linear sense for all $\boldsymbol{x} \in \Omega$. Then the SDRE nonlinear regulator produces a closed-loop solution that is locally asymptotically stable.

For the current system the controllability matrix is given by

$$[\boldsymbol{b}, \boldsymbol{A}\boldsymbol{b}, \boldsymbol{A}^2\boldsymbol{b}, \boldsymbol{A}^3\boldsymbol{b}] \tag{11}$$

For the flow conditions $U = 15$ m/s and $a = -0.68$, along with the numerical values given in Table 1, the determinant simplifies to a polynomial in terms of $x_2 = \alpha$ given by

$$\begin{aligned}\det = {} & -3.8737 + 37.056\alpha - 2291\alpha^2 + 24624.3\alpha^3 \\ & - 661443\alpha^4 + 5.31 \times 10^6\alpha^5 - 9.31 \times 10^7\alpha^6 \\ & + 5.23 \times 10^8\alpha^7 - 5.27 \times 10^9\alpha^8\end{aligned} \tag{12}$$

Table 1 System parameters

Parameter	Value
b 0.135 m	m 12.387 kg
I_a 0.065	χ_α $[0.0873 - b(1.0 + a)]/b$
k_h 2844.4 N/m	c_h 27.43 N · s/m
c_a 0.036 N · s	ρ 1.225 kg/m^3
c_{l_α} 6.28	c_{l_β} 3.358
c_{m_α} $(0.5 + a)C_{l_\alpha}$	c_{m_β} − 0.635

The pointwise controllability of the system is lost whenever the preceding determinant is zero. The preceding determinant vanishes at

$$\alpha = -.02154 \pm 0.0644j, \qquad -0.014 \pm 0.0601j$$

$$0.039 \pm 0.067j, \qquad 0.04634 \pm 0.0695j$$

which is not physically possible. In practice, one should investigate the controllability in terms of the determinant of the preceding matrix for a variety of operating conditions. A more accurate measure of the controllability can be achieved by studing the eigenvalues of the preceding matrix as a function of time.

IV. State Estimator

Consider the case in which it is possible to measure only one of the states. State estimation theory is based on a stochastic theory where it is assumed that both the process and the output equations are contaminated with noise according to the following:

$$\dot{\boldsymbol{x}} = \boldsymbol{A}(\boldsymbol{x})\boldsymbol{x} + \boldsymbol{b}\beta(t) + w(t) \tag{13}$$

$$y(t) = \boldsymbol{c}\boldsymbol{x}(t) + v(t) \tag{14}$$

where v and w are Gaussian white noises with zero mean. In addition, $E[w(t)w^\top(t+\tau)] = \boldsymbol{W}\delta(t-\tau)$ and $E[v(t)v^\top(t+\tau)] = \boldsymbol{V}\delta(t-\tau)$. To design an observer based on SDRE, it is required that the system be observable both in the linear sense and in the nonlinear sense.[10,14] For the pointwise observability in the linear sense, it is required that the following matrix has a full rank for all times:

$$[\boldsymbol{c}^\top \quad (\boldsymbol{c}\boldsymbol{A})^\top \quad (\boldsymbol{c}\boldsymbol{A}^2)^\top \quad (\boldsymbol{c}\boldsymbol{A}^3)^\top] \tag{15}$$

A necessary condition for observability in the nonlinear sense is given in terms of the rank of the following matrix (Ref. 15, p. 218):

$$\text{Rank}\begin{bmatrix} d(\boldsymbol{c}\boldsymbol{x}) \\ dL_f\boldsymbol{c}\boldsymbol{x} \\ \vdots \\ dL_f^{n-1}\boldsymbol{c}\boldsymbol{x}\end{bmatrix} = n \tag{16}$$

where $d\phi(\boldsymbol{x})$ is the Jacobian of $\phi(\boldsymbol{x})$, and $L_f\phi(\boldsymbol{x})$ is the Lie derivative of $\phi(\boldsymbol{x})$ along a vector field f. This is only a necessary condition for the observer linearization problem to be solvable (Ref. 15, p. 218). The observer linearization problem seeks a coordinate transformation such that after the transformation the system is observable in the linear sense and the error dynamics are linear. If the system is observable both in the pointwise linear sense and in the nonlinear sense, it is then possible to use the SDRE as a mechanism for generating the filter gains.

If $\hat{\boldsymbol{x}}(t)$ is the online estimate of the state then the observer dynamics is given by

$$\dot{\hat{\boldsymbol{x}}} = \boldsymbol{A}(\hat{\boldsymbol{x}})\hat{\boldsymbol{x}} + \boldsymbol{b}\beta(t) + \boldsymbol{h}_0(y(t) - \boldsymbol{c}\hat{\boldsymbol{x}}(t)) \tag{17}$$

where

$$\boldsymbol{h}_0 = -\boldsymbol{P}(\hat{\boldsymbol{x}})\boldsymbol{c}^\top \boldsymbol{V}^{-1} \tag{18}$$

The matrix $\boldsymbol{P}$ is the positive definite solution of the algebraic Riccati equation

$$\boldsymbol{A}(\hat{\boldsymbol{x}})\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}^\top(\hat{\boldsymbol{x}}) - \boldsymbol{P}\boldsymbol{c}^\top\boldsymbol{V}\boldsymbol{c}\boldsymbol{P} + \boldsymbol{W} = 0 \tag{19}$$

If the system in Eqs. (13) and (14) is pointwise observable in the linear sense, that is, the matrix in Eq. (15) has a full rank for all times, then the preceding equation can be solved uniquely for the positive-definite matrix $\boldsymbol{P}$. The preceding estimator can be started from a zero initial condition, that is, $\hat{\boldsymbol{x}}(0) = 0$. Due to its construction, if the states are accurately estimated at, say, $t = \tau$, then the estimator follows the states for all times greater than τ.

V. Performance Studies

In general, not all of the states are available online and the feedback law must be based on an estimate of the states. Consider the case where it is possible to measure α as a function of time. In other words, the output equation is given by

$$y(t) = \boldsymbol{c}\boldsymbol{x}(t), \qquad \boldsymbol{c} = [0, 1, 0, 0] \tag{20}$$

For this case the pointwise observability matrix in Eq. (15) is given by

$$\begin{bmatrix} 0 & 0 & -k_3 & c_3k_1 + c_4k_3 \\ 1 & 0 & -k_4U^2 - q(x_2) & c_3\left[k_2U^2 + p(x_2)\right] + c_4\left[k_4U^2 + q(x_2)\right] \\ 0 & 0 & -c_3 & -k_3 + c_1c_3 + c_3c_4 \\ 0 & 1 & -c_4 & -k_4U^2 - q(x_2) + c_2c_3 + c_4^2 \end{bmatrix} \tag{21}$$

The determinant of the preceding matrix is given by $-k_3^2 + c_1c_3k_3 - c_3^2k_4$, which, for the specific data given in Table 1, is equal to $-8{,}819{,}752.43 (\neq 0)$. Therefore, the preceding matrix has a full rank and it follows that the system is pointwise observable in the linear sense. A necessary condition for observability in the nonlinear sense is given in terms of the rank of the matrix in Eq. (16). This is only a necessary condition for the observer linearization problem to be solvable (Ref. 15, p. 218). For the nonlinear system studied here, the preceding matrix simplifies to

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -k_3 & -k_4(U^2 + q(x_2)) & -c_3 & -c_4 \\ c_3k_1 + c_4k_3 & -q'(x_2)x_4 + c_3\left[k_2U^2 + p(x_2)\right] + c_4\left[k_4U^2 + q(x_2)\right] & -k_3 + c_3(c_1 + c_4) & -\left[k_4U^2 + q(x_2)\right] + c_3c_2 + c_4^2 \end{bmatrix} \tag{22}$$

where $q'(x_2) = \partial q(x_2)/\partial x_2$. The determinant of the preceding matrix is given by $k_3^2 - c_1c_3k_3 + c_3^2k_1$, which, for the specific data given in Table 1, is equal to 9,322,867.77. The preceding matrix has a full rank for all times and, therefore, it is possible to design an on-line observer.

Consider the case in which $U = 15$ m/s and $a = -0.68$ and which starts from the initial conditions $\alpha = 0.1$ rad and $h = 0.01$ m. For the measurement noise variance $\boldsymbol{V} = 1E - 4$ and $\boldsymbol{W} = \boldsymbol{I}$, where $\boldsymbol{I}$ is the 4×4 identity matrix, the performance of the observer is given in Fig. 2. In Fig. 2, the variables $h(t)$ and $\dot{\alpha}(t)$ are scaled to fit them in the same plot. After some initial transients the estimator can accurately recover the states. These estimates can then be used in a feedback control law.

We can also consider the partial observation case where, for a single-input system studied in this paper, the optimal control problem is based on the cost functional. Minimize

$$J = \frac{1}{2}\int_0^\infty [\boldsymbol{x}^\top\boldsymbol{c}^\top\boldsymbol{c}\boldsymbol{x} + ru^2]\,\mathrm{d}t \tag{23}$$

The feedback control law is given by Eq. (10), where the symmetric matrix $\boldsymbol{P} \geq 0$ is the solution of the algebraic Riccati equation (ARE):

$$\boldsymbol{A}^\top(\boldsymbol{x})\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}(\boldsymbol{x}) - \boldsymbol{P}\boldsymbol{b}r^{-1}\boldsymbol{b}^\top\boldsymbol{P} + \boldsymbol{c}^\top\boldsymbol{c} = 0 \tag{24}$$

Figure 3 shows the closed-loop response of the system, where, again, the variables $h(t)$ and $\beta(t)$ are scaled. If the feedback law is based on the estimate of the states then this constitutes a compensator for the system. If the feedback law seeks to minimize a partial observation cost functional given in Eq. (24), then no online observer is necessary. In both cases the feedback laws can effectively control the flutter. For this case a compensator leads to a less oscillatory closed-loop response, which is more desirable. Figure 3 also shows the feedback control law $\beta(t)$. The variables $x_3 = \dot{h}(t)$ and $x_4 = \dot{\alpha}(t)$, which are not shown, and are the time derivatives of the plunge $h(t)$ and α, also decay to zero. Figure 4 shows the response of the system

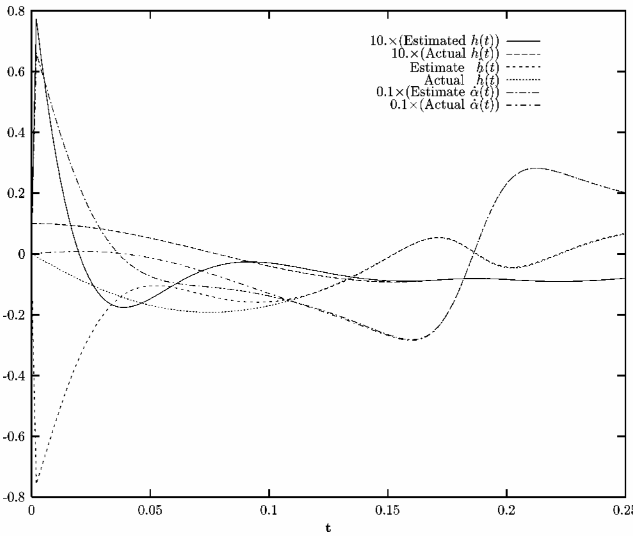

Fig. 2 Performance of the estimator as a function of time. The variables $h(t)$ and $\dot{\alpha}(t)$ are scaled to fit in the same plot. The measured variable is $\alpha(t)$. The units for the variables are radians (α) and meters [$h(t)$].

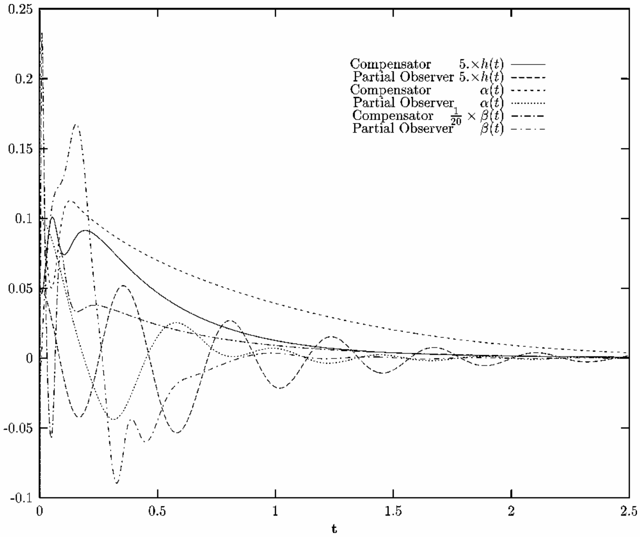

Fig. 3 Performance of the feedback law as a function of time when it is possible to measure $\alpha(t)$. The value of the plunge $h(t)$ and the control $\beta(t)$ are scaled to fit in the same plot. The units for the variables are radians (α) and meters [$h(t)$].

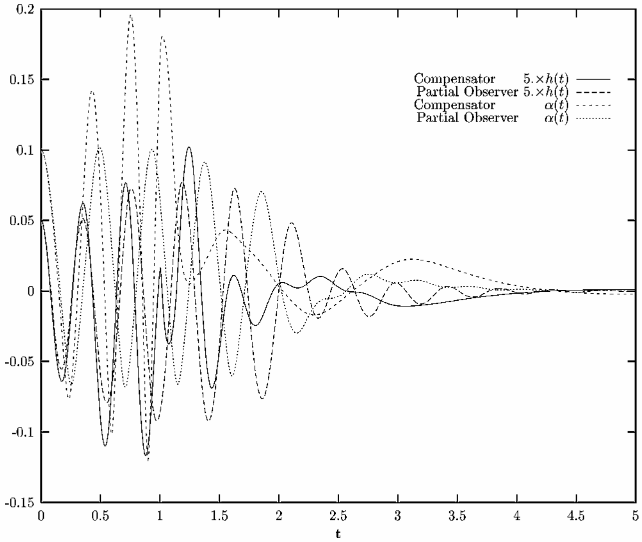

Fig. 4 Performance of the feedback law as a function of time when it is possible to measure $h(t)$. The value of the plunge $h(t)$ is scaled to fit in the same plot. The units for the variables are radians (α) and meters [$h(t)$].

when the plunge $h(t)$ is measured as a function of time. In a similar way one can check the pointwise observability of the system for this case. The closed-loop response for this case indicates that the two feedback laws lead to closed-loop responses that are similar in behavior. For this case a feedback law based on partial observation may be more desirable due to a less computational effort.

VI. Conclusions

We studied the effectiveness of a compensator in controlling an aeroelastic system. The feedback control system makes use of one online measurement and generates feedback laws. Two similar feedback laws can be constructed. The benefits of using one vs the other depend in part on the state that is being measured online. In both cases the algorithm leads to a fast closed-loop response, which is desirable.

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